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Shock Propagation Produced by Planar Impact in Linearly Elastic Anisotropic Media*

J. N. Johnson
Sandia Laboratories, Albuquerque, New Mexico 87115
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Plate-impact experiments and shock-wave propagation in isotropic materials have become a standard technique for the study of material properties at very high strain rates. Recent work has also included shock propagation in anisotropic materials for specific wave-propagation directions in which pure longitudinal motion can exist. In this work, the effect of anisotropic elasticity and its influence on plane-wave propagation for arbitrary loading directions in plate-impact experiments is studied. The analysis applies specifically to the propagation of elastic disturbances produced by planar impact in a series of anisotropic plates of dissimilar materials each of arbitrary crystallographic orientation. The boundary conditions that exist between adjacent materials as well as conditions at free surfaces are considered in detail. It is found that for general orientations of single-crystal Be and Ti (hexagonal symmetry), multiple-wave effects, and transverse particle motion resulting from the elastic anisotropy are small. However, for y-cut α quartz (trigonal symmetry) these effects are significant. Transverse velocity components have been found to be greater than 35% of the maximum longitudinal velocity component.

I. INTRODUCTION

Plate-impact experiments provide a very useful technique for the study of dynamic material properties at high strain rates. Most of the research work carried out in this area has been performed on isotropic materials. For such materials the analysis of the experimental data is simplified because conditions of uniaxial strain exist at the position of measurement during times of experimental observation. The use of the plate-impact experiment in studies of isotropic materials has been reviewed by Karnes,[1] Gilman,[2] and Herrmann.[3] Recently, a number of anisotropic single crystals[4–7] have been studied under conditions of shock-wave compression, but in all cases the shock-loading direction was chosen to be one in which pure longitudinal-elastic disturbances could be propagated as discussed by Borgnis[8] and Brugger.[9] The question then arises regarding what happens when linearly elastic anisotropic materials are shock loaded in directions other than the specific ones which result in pure longitudinal motion. While in principle the ideal condition of a single purely longitudinal wave may not be realized for arbitrary orientation, in some cases anisotropic effects may be small and it is of interest to ascertain the magnitude of the departure from this condition.

In isotropic materials, transverse motion in a plane wave is produced only by the application of shear stresses at the boundary. For example, the application of a shear stress across a plane perpendicular to the propagation direction will result in transverse motion. In anisotropic materials, transverse motion can be produced by the coupling that exists between all components of stress and strain. For some materials, the degree of coupling may be small, either by virtue of a significant amount of symmetry possessed by the crystal or through the magnitude of the elastic constants being nearly representative of an isotropic material.

To study this problem, the classical equations of linear elasticity are applied to the case of stress-wave propagation generated by planar impact in a series of anisotropic plates of dissimilar materials each of arbitrary crystallographic orientation.[10] The solutions, which involve discontinuous changes in stress, particle velocity, etc., thus approximate the propagation of shock waves in real materials subject to finite strain compression. This work presents no new results for research workers in the area of ultrasonics and the determination of elastic moduli, but is expected to be of considerable interest to those studying material properties and wave-propagation effects under conditions of planar impact.

II. GENERAL DEVELOPMENT

The propagation of acoustic waves in linear elastic homogeneous media is governed by the expression [11,12]

$$\rho \frac{\partial^2 u_i}{\partial t^2} = C_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} , \qquad (1)$$

where ρ is the material density, C_{ijkl} is the fourth-order elastic-constant tensor, and u_i is the material displacement in the direction defined by the Cartesian coordinates $x_i (i=1, 2, 3)$. Throughout this work, the coordinates x_i are chosen to correspond to the crystallographic coordinate system and t is the time. For plane waves propagating in a direction defined by the unit vector $\vec{a}$, a solution to Eq. (1) can be written in the form

$$u_i = U_i f(t - a_m x_m / v), \tag{2}$$

where f is some arbitrary function, U_i is the relative-displacement amplitude in the three Cartesian directions, and v is the wave velocity.[13] Substitution of Eq. (2) into Eq. (1) yields the following expression relating the relative-displacement amplitudes:

$$(\Gamma_{ik} - \delta_{ik}\rho v^2) U_k = 0, \tag{3}$$

where δ_{ik} is the Kronecker δ and the acoustic tensor Γ_{ik} is defined by[11]

$$\Gamma_{ik} \equiv C_{ijkl} a_j a_l. \tag{4}$$

Equation (3) gives the relationship between the elastic constants, the propagation direction, the wave velocity, and the relative-displacement amplitudes in the three Cartesian-coordinate directions. It can be seen that nontrivial solutions for the U_k in Eq. (3) are obtained only if the determinant of coefficients, i.e., $\det(\Gamma_{ik} - \delta_{ik}\rho v^2)$, is equal to zero. This puts a restriction on the allowable values of ρv^2; namely, ρv^2 must equal one of the three eigenvalues of the acoustic tensor Γ_{ik}. For ρv^2 equal to a particular eigenvalue, the relative-displacement vector $\vec{U}$ is equal to the corresponding eigenvector. For example, in an isotropic solid with $\vec{a} = (1, 0, 0)$ the three allowable values of ρv^2 are $\frac{1}{2}(C_{11} - C_{12})$, $\frac{1}{2}(C_{11} - C_{12})$, and C_{11} while the corresponding values of $\vec{U}$ are $\vec{U}^{(1)} = (0, 1, 0,)$, $\vec{U}^{(2)} = (0, 0, 1)$, and $\vec{U}^{(3)} = (1, 0, 0)$, respectively. These values of ρv^2 and $\vec{U}$ represent two pure shear waves and one pure longitudinal wave. In a general anisotropic medium, the three relative-displacement amplitudes $\vec{U}^{(1)}$, $\vec{U}^{(2)}$, and $\vec{U}^{(3)}$ are always mutually orthogonal, but their directions are usually neither orthogonal to nor coincident with the propagation direction $\vec{a}$. Waves for which particle motion is predominantly perpendicular or parallel to the propagation direction are therefore termed quasitransverse (QT) or quasilongitudinal (QL), respectively. The three allowable waves in an anisotropic material generally consist of two QT waves and one QL wave.

Values of ρv_i^2 and $\vec{U}^{(i)}, i = 1, 2, 3$, have been calculated for a number of specific cases. The results are given in Tables I–III. For wave propagation in materials of cubic symmetry, these quantities can be expressed in closed form for wave-propagation vectors lying in either the $\{100\}$ or $\{110\}$ planes.

TABLE I. Eigenvalues and eigenvectors of the acoustic tensor for wave-propagation vectors lying in $\{100\}$ and $\{110\}$ planes in materials of cubic symmetry.

$\vec{a}$[a]	i	λ_i	$\vec{U}^{(i)}$
(100) plane	1(T)	0	(0, 0, 1)
$\vec{a} = (\cos\varphi, \sin\varphi, 0)$	2(QT)	$\frac{1}{2}(\alpha - R)$	$(-\Psi, 1, 0)$
	3(QL)	$\frac{1}{2}(\alpha + R)$	$(1, \Psi, 0)$
(110) plane	1(T)	$[\frac{1}{2}(1-\alpha)] \sin^2\theta$	$(1, -1, 0)$
$\vec{a} = (a_1, a_2, \cos\theta)$	2(QT)	λ_-	$(1, 1, U_-)$
	3(QL)	λ_+	$(1, 1, U_+)$

[a] For a wave-propagation vector lying in the (100) plane, φ is the angle between x_1' and x_1. For a wave-propagation vector lying in the (110) plane, θ is the angle between x_1' and x_3: Here $a_1 = a_2 = (\frac{1}{2})^{1/2} \sin\theta$.

These results are given in Table I. The more general case has been discussed by Miller *et al.*[14] Table II gives values of ρv_i^2 and $\vec{U}^{(i)}$ for an arbitrary propagation direction in crystals of hexagonal symmetry. Table III gives these quantities for materials of trigonal symmetry for $\vec{a} = (1, 0, 0)$ and $(0, 1, 0)$. These results thus apply to x-cut and y-cut α quartz, respectively. Additional information corresponding to these particular cases as well as for other crystal symmetries and propagation directions is given by Hearmon.[15] The parameters used in the expressions for the eigenvalues and eigenvectors in Tables I–III are defined as follows:

Table I:

$$\alpha = (C_{11} - C_{44})/(C_{12} + C_{44}), \quad \rho v_i^2 = (C_{12} + C_{44})\lambda_i + C_{44},$$

$$R = [\alpha^2 \cos^2 2\varphi + \sin^2 2\varphi]^{1/2}, \quad \Psi = (R - \alpha\cos 2\varphi)/\sin 2\varphi,$$

$$\lambda_\pm = (\tfrac{1}{2}\alpha) + [\tfrac{1}{4}(1-\alpha)]\sin^2\theta \pm \tfrac{1}{2}\{[\tfrac{1}{2}(3\alpha+1)\sin^2\theta - \alpha]^2 - \sin^2 2\theta\}^{1/2},$$

$$2^{1/2} U_\pm \sin\theta\cos\theta = \alpha - \tfrac{1}{2}(3\alpha+1)\sin^2\theta \pm \{[\tfrac{1}{2}(3\alpha+1)\sin^2\theta - \alpha]^2 - \sin^2 2\theta\}^{1/2};$$

Table II:

$$\rho v_1^2 = \tfrac{1}{2}(C_{11} - C_{12})\sin^2\theta + C_{44}\cos^2\theta,$$

$$2\rho v_\pm^2 = C_{11}\sin^2\theta + C_{33}\cos^2\theta + C_{44} \pm S,$$

$$\chi_\pm = [(C_{13} + C_{44})\sin 2\theta]^{-1} \times [C_{11}\sin^2\theta - C_{33}\cos^2\theta + C_{44}\cos 2\theta \pm S],$$

$$S^2 = (C_{11}\sin^2\theta - C_{33}\cos^2\theta + C_{44}\cos 2\theta)^2 + (C_{13} + C_{44})^2 \sin^2 2\theta;$$

Table III:

$$\rho v_\pm^2 = \tfrac{1}{2}[C_{44} + \tfrac{1}{2}(C_{11} - C_{12})] \pm r,$$

$$4r^2 = [C_{44} - \tfrac{1}{2}(C_{11} - C_{12})]^2 + 4C_{14}^2,$$

$$2C_{14}\zeta = C_{44} - \tfrac{1}{2}(C_{11} - C_{12}) + 2r,$$

$$C_{14}\xi = -R + \tfrac{1}{2}(C_{11} - C_{44}), \quad 4R^2 = (C_{11} - C_{44})^2 + 4C_{14}^2 .$$

Let us now consider the case of planar impact between two different anisotropic plates of infinite lateral extent. One plate is assumed to be traveling with velocity V_0 in the positive x_1' direction,[16] while the other is at rest: The impact faces of these plates are parallel to the x_2', x_3' plane. Planar impact between these two materials at $t=0$ and $x_1'=0$ results in a maximum of three disturbances traveling to the left into material 1 and three disturbances traveling to the right into material 2 as shown in the x_1', t plane in Fig. 1. In each material these waves are discontinuous jumps in material velocity traveling with speeds equal to the allowable velocities v_i. If we denote the flow variables and parameters in the nth material by the presuperscript n, a solution which satisfies Eq. (1) and the initial conditions in the two materials can be written

$$^1\vec{u}' = \sum_{i=1}^{3} {}^1\beta_i \, {}^1\vec{U}^{(i)\prime} h(t + x_1'/{}^1v_i) + (V_0 t, 0, 0), \qquad (5)$$

$$^2\vec{u}' = \sum_{i=1}^{3} {}^2\alpha_i \, {}^2\vec{U}^{(i)\prime} h(t - x_1'/{}^2v_i) , \qquad (6)$$

where $h(\tau) \equiv \int_{-\infty}^{\tau} H(z)\,dz$, $H(z)$ is the Heaviside step function, and the primes indicate that the vector components are with respect to the x_1', x_2', x_3' coordinate system. Tensor quantities in this coordinate system are related to quantities in the crystallographic-coordinate system through transformation matrix

$$A_{kl} = \vec{i}_k' \cdot \vec{i}_l , \qquad (7)$$

where $\vec{i}_k'$ and $\vec{i}_l$ are unit vectors in the x_k' and x_l directions, respectively. The six quantities $^1\beta_i$, $^2\alpha_i$ $(i=1,2,3)$ are constants determined by the boundary conditions at the impact surface.

TABLE II. Eigenvalues and eigenvectors of the acoustic tensor for wave propagation in an arbitrary direction in materials of hexagonal symmetry.

$\vec{a}$ [a]	i	λ_i	$\vec{U}^{(i)}$
	1(T)	ρv_1^2	(1, 0, 0)
(0, sinθ, cosθ)	2(QT)	ρv_-^2	$(0, \chi_-, 1)$
	3(QL)	ρv_+^2	$(0, \chi_+, 1)$

[a] There is no loss of generality in choosing $a_1 = 0$, since all directions perpendicular to the x_3 axis are equivalent. Here θ is the angle between x_1' and x_3.

TABLE III. Eigenvalues and eigenvectors of the acoustic tensor for [100] and [010] propagation in materials of trigonal symmetry.

$\vec{a}$	i	ρv_i^2	$\vec{U}^{(i)}$
	1(T)	ρv_-^2	$(0, -\zeta, 1)$
(1, 0, 0)	2(T)	ρv_+^2	$(0, 1, \zeta)$
	3(L)	C_{11}	(1, 0, 0)
	1(T)	$\frac{1}{2}(C_{11} - C_{12})$	(1, 0, 0)
(0, 1, 0)	2(QT)	$[\frac{1}{2}(C_{11} + C_{44})] - R$	$(0, -\xi, 1)$
	3(QL)	$[\frac{1}{2}(C_{11} + C_{44})] - R$	$(0, 1, \xi)$

The three nonzero components of strain $\epsilon_{ij}' \equiv -\frac{1}{2}(\partial u_i'/\partial x_j' + \partial u_j'/\partial x_i')$ in the x_1', x_2', x_3' coordinate system are ϵ_{11}, ϵ_{12}, and ϵ_{13}. If we define the square matrix of elastic constants C as

$$\underline{C} \equiv \begin{bmatrix} C_{11}' & C_{16}' & C_{15}' \\ C_{16}' & C_{66}' & C_{56}' \\ C_{15}' & C_{56}' & C_{55}' \end{bmatrix} , \qquad (8)$$

the three components of stress σ_{11}', σ_{12}', and σ_{13}' are given by

$$\vec{s} = \underline{C}\,\vec{e}, \qquad (9)$$

$$\vec{s} \equiv \begin{bmatrix} \sigma_{11}' \\ \sigma_{12}' \\ \sigma_{13}' \end{bmatrix}, \text{ and } \vec{e} \equiv \begin{bmatrix} \epsilon_{11}' \\ 2\epsilon_{12}' \\ 2\epsilon_{13}' \end{bmatrix} . \qquad (10)$$

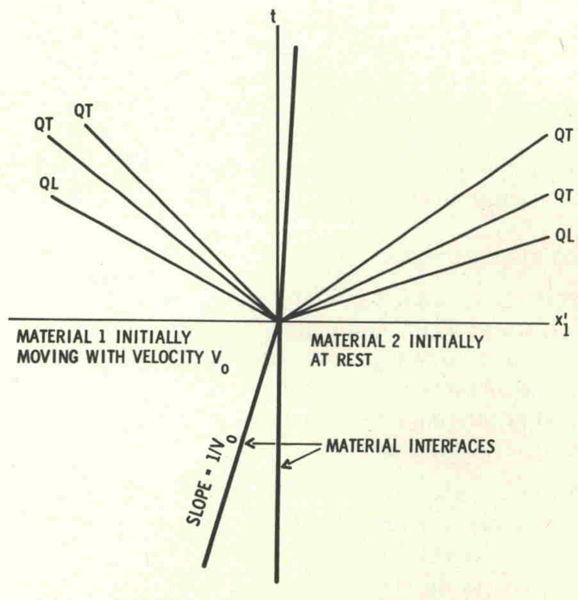

FIG. 1. Impact between two anisotropic materials of arbitrary orientation is shown in the x_1', t plane. Three waves traveling in each direction are generated at the impact surface. Two QT waves and one QL wave are produced.

From Eqs. (5) and (6) the strain in materials 1 and 2 gives the following expressions for the components of the column matrix $\vec{e}$ at the impact surface:

$$^{2}e_m = -\sum_{i=1}^{3} {}^{1}(\beta_i/v_i)\, {}^{1}U_m^{(i)\prime}, \quad (11)$$

$$^{2}e_m = \sum_{i=1}^{3} {}^{2}(\alpha_i/v_i)\, {}^{2}U_m^{(i)\prime}. \quad (12)$$

From Eqs. (8)–(12) and the assumption that the matrix $\vec{s}$ must be continuous across the impact surface, i.e.,

$$^{1}\vec{s} = {}^{2}\vec{s}, \quad (13)$$

we obtain three equations in terms of $^{2}\alpha_i$ and $^{1}\beta_i$ $(i=1,2,3)$. An additional requirement on the solution is that the u_1' be continuous across the impact surface:

$$^{1}u_1' = {}^{2}u_1'. \quad (14)$$

To complete the description of the boundary conditions at the interface between the two materials, we must specify the degree to which this interface can support a shear stress. The two extreme cases are (i) the boundary is perfectly smooth and can support no shear stresses, or (ii) upon impact the two materials stick together and all displacement components are continuous across the interface. These two extremes, plus the continuum of cases in between, can be represented by the following expressions:

$$^{1}\sigma_{12}' = \eta({}^{1}\dot{u}_2' - {}^{2}\dot{u}_2'), \quad (15)$$

$$^{1}\sigma_{13}' = \eta({}^{1}\dot{u}_3' - {}^{2}\dot{u}_3'), \quad (16)$$

where η is a constant. Equations (15) and (16) represent a linearized law of sliding friction in which the shear stress transmitted across the interface is proportional to the difference in velocity between the two materials. For $\eta = 0$, the interface transmits zero shear stress; for $(1/\eta) = 0$, the displacements are continuous.

Therefore, Eqs. (13)–(16) provide six boundary conditions which determine the constants $^{2}\alpha_i$ and $^{1}\beta_i (i=1,2,3)$ in Eqs. (5) and (6). The solution that is obtained is applicable until the waves generated at the impact surface reach another boundary and reflections occur.

From the special case of planar impact between two dissimilar anisotropic materials that we have just discussed, it is easy to write a more general solution to Eq. (1) which is applicable to the propagation of a series of stress and particle-velocity discontinuities produced by planar impact and subsequent wave reflections at material interfaces. We consider a series of plates of infinite lateral extent. The location of the nth plate is specified by the positions of its left and right boundaries L_n and R_n, respectively. The material making up a given plate, its crystal symmetry, and its orientation are assumed to be arbitrary. For every wave interaction at a particular interface, there exists the possibility that this leads to three waves traveling into the material on the right-hand side and three waves traveling into the material on the left-hand side of the boundary. Hence, in the nth layer the velocity is given by

$$^{n}\dot{\vec{u}}' = \sum_{j=1}^{\infty}\sum_{i=1}^{3} {}^{n}\alpha_{ij}\, {}^{n}\vec{U}^{(i)\prime} H\left(t - {}^{n}t_j - \frac{x_1' - L_n}{{}^{n}v_i}\right) + \sum_{k=1}^{\infty}\sum_{i=1}^{3} {}^{n}\beta_{ik}\, {}^{n}\vec{U}^{(i)\prime} H\left(t - {}^{n}t_k + \frac{x_1' - R_n}{{}^{n}v_i}\right) + (V_n,\ 0,\ 0), \quad (17)$$

where V_n is the initial velocity of the nth material, $^{n}t_j$ is the time of the jth interaction at L_n, $^{n}t_k$ is the time of the kth interaction at R_n, and the constants $^{n}\alpha_{ij}$ and $^{n}\beta_{ik}$ are determined by the boundary conditions at the material interfaces. The components of the column matrix $\vec{e}$ defined by Eq. (10) are

$$^{n}e_m(x_1', t) = \sum_{j=1}^{\infty}\sum_{i=1}^{3} \frac{^{n}\alpha_{ij}}{^{n}v_i}\, {}^{n}U_m^{(i)\prime} H\left(t - {}^{n}t_j - \frac{x_1' - L_n}{{}^{n}v_i}\right) - \sum_{k=1}^{\infty}\sum_{i=1}^{3} \frac{^{n}\beta_{ik}}{^{n}v_i}\, {}^{n}U_m^{(i)\prime} H\left(t - {}^{n}t_k + \frac{x_1' - L_n}{{}^{n}v_i}\right). \quad (18)$$

Substitution of Eq. (18) into Eq. (9) gives the stress in the nth plate as a function of x_1' and t. In analogy to Eqs. (13)–(16), the boundary conditions at the interface between the nth and $(n+1)$th plates are given by

$$^{n}\vec{s}(R_n,\ t) = {}^{n+1}\vec{s}(L_{n+1},\ t), \quad (19)$$

$$^{n}u_1'(R_n, t) = {}^{n+1}u_1'(L_{n+1}, t), \quad (20)$$

$$^{n}\sigma_{12}'(R_n, t) = \eta[{}^{n}\dot{u}_2'(R_n, t) - {}^{n+1}\dot{u}_2'(L_{n+1}, t)], \quad (21)$$

$$^{n}\sigma_{13}'(R_n, t) = \eta[{}^{n}\dot{u}_3'(R_n, t) - {}^{n+1}\dot{u}_3'(L_{n+1}, t)]. \quad (22)$$

These boundary conditions determine the six constants $^{n}\beta_{ik}$ and $^{n+1}\alpha_{ij}$ arising from the jth interaction at $x_1' = L_{n+1}$ (this is equivalent to the kth interaction at $x_1' = R_n$, since $L_{n+1} = R_n$, $^{n+1}t_j = {}^{n}t_k$, and $j = k$). Equations (17)–(22) thus provide solutions to Eq. (1) for a wide variety of problems involving planar impact in series of anistropic plates.

While Eqs. (19)–(22) represent the boundary conditions at the interface between two materials, it is of particular interest to consider as a special case the interaction of an oncoming stress or particle-velocity discontinuity with a free surface. We consider a single wave of type r which interacts with a free surface at time t_1 and position $x_1' = R$ as shown in Fig. 2. The interaction of this wave with

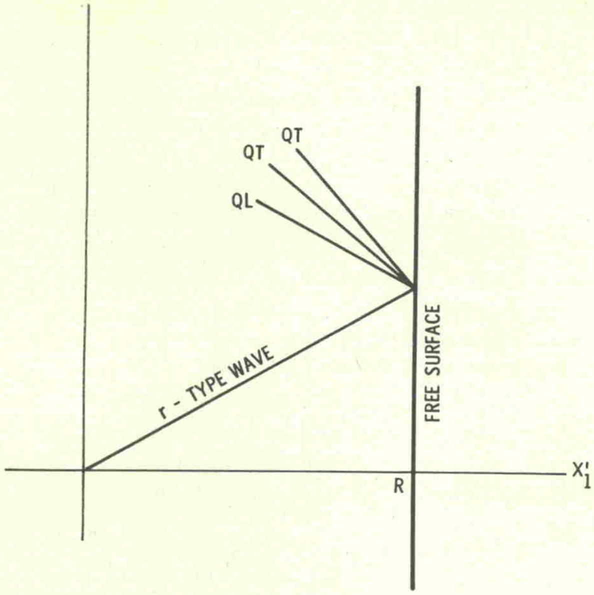

FIG. 2. An r-type wave ($r=1$, 2, or 3 depending on which of the two QT or single QL waves are considered) is incident on the free surface at $x_1'=R$. In general, one would expect that three waves could be reflected. However, it can be shown that only a single r-type wave is reflected.

the free surface is assumed to result in three waves transmitted back into the sample. Thus, we write a solution to Eq. (1) as[17]

$$\dot{\vec{u}}' = \alpha_r \vec{U}^{(r)\prime} H(t - x_1'/v_r) + \sum_{i=1}^{3} \beta_i \vec{U}^{(i)\prime} H[t - t_1 + (x_1' - R)/v_i]. \tag{23}$$

The components of $\vec{e}$ defined by Eq. (10) are

$$e_m = \sum_{i=1}^{3} [(\alpha_r/v_r)\delta_{ir} - \beta_i/v_i]\, U_m^{(i)\prime} \tag{24}$$

for $t > t_1$ at the free surface. Since this is a stress-free boundary, Eq. (9) requires that

$$\underline{C}\vec{e} = 0 \tag{25}$$

or

$$\underline{C}\,\underline{D}\,\vec{\gamma} = 0, \tag{26}$$

where

$$D_{mi} \equiv U_m^{(i)\prime} \tag{27}$$

and

$$\gamma_i \equiv \alpha_r/v_r\ \delta_{ir} - \beta_i/v_i\,. \tag{28}$$

Equation (25) is a homogeneous set of equations for $\gamma_i (i = 1, 2, 3)$ which has nontrivial solutions only if $\det(\underline{C}\cdot\underline{D}) = 0$. Since we have

$$\det(\underline{C}\cdot\underline{D}) = (\vec{U}^{(1)\prime}\cdot\vec{U}^{(2)\prime} \times \vec{U}^{(3)\prime}) \cdot \det(\underline{C}),$$

the only solution to Eq. (25) is the trivial one when we have

$$\det(\underline{C}) \neq 0.$$

Therefore, $\beta_i = \alpha_r \delta_{ir}$ for reflection from a free boundary. This simply means that the interaction of a type r wave with a free boundary results in a reflected wave of the same type and with a displacement amplitude equal to that of the incident wave.

III. SPECIFIC APPLICATIONS

In Sec. II, a somewhat general analysis of plane-wave propagation resulting from planar impact of linearly elastic anisotropic materials was presented. In this section, a number of specific applications are worked out in some detail. The first to be considered is plane-wave propagation in Be and Ti (hexagonal symmetry) for arbitrary crystallographic orientation, and the second case is planar impact between x-cut and y-cut quartz plates (trigonal symmetry). The densities and elastic constants used in these calculations are given in Table IV.[18-20]

Hexagonal Symmetry

Eigenvalues and eigenvectors of the acoustic tensor for wave propagation in an arbitrary direction in materials of hexagonal symmetry are given in Table II. In the example considered here, the outward normal of the loaded surface is the negative of the wave-propagation direction $\vec{a} = (0, \sin\theta, \cos\theta)$: Since all directions perpendicular to x_3 are equivalent for hexagonal symmetry, a_1 is chosen zero without loss of generality. The sample material is assumed to be semi-infinite in the direction of wave propagation; this is the same as saying that a solution is sought for times before wave interactions occur at interfaces that may exist in an actual physical situation. A propagating disturbance is initiated at the boundary by the application of stresses given by

$$\vec{s} = \begin{bmatrix} \sigma_0 \\ 0 \\ 0 \end{bmatrix}, \tag{29}$$

where $\vec{s}$ is defined by Eq. (10). This specification of the boundary condition appears to differ some-

TABLE IV. Densities (g/cm^3) and elastic constants (kbar) for Be, Ti, and α quartz as given in Refs. 18–20.

Material	ρ	C_{11}	C_{12}	C_{13}	C_{14}	C_{33}	C_{44}
Be	1.85	2923	267	140	0	3364	1625
Ti	4.51	1624	920	690	0	1807	467
Quartz	2.65	868	70	119	-180	1058	582

what from the general analysis given in Sec. II in which the stress wave is initiated by the impact between two materials. However, if one considers impact between two materials that will not support a shear stress across the boundary, i.e., $\eta = 0$ in Eqs. (15) and (16), then Eq. (29) correctly represents these conditions. Here σ_0 is related to the impact velocity and the material properties of the impactor as well as those of the sample being impacted. Thus, when one is interested only in waves propagating in one material, the boundary conditions described by Eq. (29) are sufficient for the determination of the constants α_1, α_2, and α_3 in the expression for the particle velocity

$$\dot{\vec{u}}' = \sum_{i=1}^{3} \alpha_i \vec{U}^{(i)\prime} H(t - x_1'/v_i). \quad (30)$$

The boundary conditions are satisfied when we have

$$\frac{\alpha_1}{v_1} = 0, \quad \frac{\alpha_2}{v_2} = \frac{\sigma_0 B_{22}}{D}, \quad \frac{\alpha_3}{v_3} = \frac{-\sigma_0 B_{21}}{D}, \quad (31)$$

where $v_2 = v_-$, $v_3 = v_+$; χ_- and χ_+ are defined in Table II and

$$D = B_{11}B_{22} - B_{12}B_{21}, \quad (32)$$

$$B_{11} = C_{11}'(\chi_- \sin\theta + \cos\theta) + C_{15}'(\chi_- \cos\theta - \sin\theta), \quad (33)$$

$$B_{12} = C_{11}'(\chi_+ \sin\theta + \cos\theta) + C_{15}'(\chi_+ \cos\theta - \sin\theta), \quad (34)$$

$$B_{21} = C_{15}'(\chi_- \sin\theta + \cos\theta) + C_{55}'(\chi_- \cos\theta - \sin\theta), \quad (35)$$

$$B_{22} = C_{15}'(\chi_+ \sin\theta + \cos\theta) + C_{55}'(\chi_+ \cos\theta - \sin\theta), \quad (36)$$

$$C_{11}' = C_{11}\sin^4\theta + C_{33}\cos^4\theta + \tfrac{1}{2}(C_{13} + 2C_{44})\sin^2 2\theta, \quad (37)$$

$$C_{15}' = \sin\theta\cos\theta \times [C_{11}\sin^2\theta - C_{33}\cos^2\theta + (C_{13} + 2C_{44})\cos 2\theta], \quad (38)$$

$$C_{55}' = (C_{11} + C_{33} - 2C_{13})\sin^2\theta\cos^2\theta + C_{44}\cos^2 2\theta. \quad (39)$$

The transformed eigenvectors $\vec{U}^{(i)\prime}$ are obtained from the $\vec{U}^{(i)}$ given in Table II according to the relationship

$$U_k^{(i)} = A_{kl}\, U_l^{(i)}, \quad (40)$$

where A_{kl} is defined by Eq. (7) and in this particular case is taken as

$$\underline{A} = \begin{bmatrix} 0 & \sin\theta & \cos\theta \\ 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \end{bmatrix}. \quad (41)$$

From these results, the component of stress in the direction of propagation is then given by

$$\sigma_{11}' = (\sigma_0/D)[B_{11}B_{22}H(t - x_1'/v_2) - B_{12}B_{21}H(t - x_1'/v_3)] \quad (42)$$

The first term in the brackets on the right-hand side of Eq. (42) represents a quasitransverse wave propagating with velocity v_2 and the second term represents a quasilongitudinal wave propagating with velocity $v_3 > v_2$. The longitudinal component of stress behind the leading disturbance is

$$\sigma_1 = -\sigma_0 B_{12}B_{21}/D, \quad (43)$$

while the stress behind the second disturbance is simply σ_0. Thus a measure of departure from pure longitudinal wave propagation, in which a single wave of amplitude σ_0 would exist, is $1 - \sigma_1/\sigma_0$. This quantity is plotted in Fig. 3 as a function of θ for two materials with hexagonal crystal structure; Be and Ti. For $\theta = 0°$ and 90°, $1 - \sigma_1/\sigma_0 = 0$ as expected.[8] For Be the quantity $1 - \sigma_1/\sigma_0$ also vanishes for $\theta = 13.3°$[21] and reaches a maximum of approximately 0.8% at $\theta = 65°$. For Ti the maximum value of $1 - \sigma_1/\sigma_0$ is considerably smaller at 0.25% for $\theta = 30°$.

From the results of these calculations it can be seen that although multiple waves are propagated for $\theta \neq 0°$ and 90°, the departure from the case of pure longitudinal wave propagation is small for Be and Ti. This is especially true in the vicinity of $\theta = 0°$ for beryllium and near $\theta = 90°$ in the case of titanium.

Trigonal Symmetry

The second example representing application of the general analysis is the repsonse of a y-cut quartz plate which is impact loaded with a very thick x-cut quartz plate. In this example, interactions of quasitransverse and quasilongitudinal waves with a free surface are considered in addition to effects resulting from different values of η in the boundary conditions given by Eqs. (15) and (16).

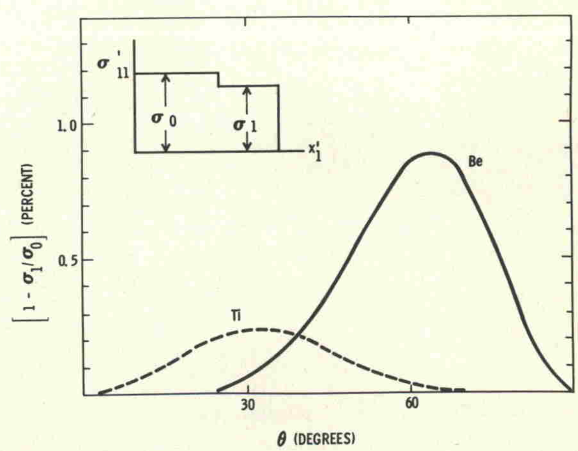

FIG. 3. The quantity $1 - \sigma_1/\sigma_0$ is shown as a function of the angle θ for application of the boundary condition $\sigma_{11}' = \sigma_0$ in Be and Ti.

The geometry in this example is shown in Fig. 4. The x-cut quartz plate moving with an initial velocity V_0 is termed material 1 and all variables associated with this material carry the appropriate presuperscript. The y-cut quartz plate of thickness L is termed material 2. Before wave interactions occur at the free surface located $x_1'=L$, the displacement vectors in both materials are given by Eqs. (5) and (6). Analytical expressions for ${}^1\alpha_i$ and ${}^2\beta_i$ $(i=1,2,3)$ are most easily obtained for $\eta=0$ in Eqs. (15) and (16); in this case, we have

$$ {}^2\alpha_1=0, \quad {}^1\beta_1=0, \quad {}^1\beta_2=0, \tag{44}$$

$$ {}^2\alpha_2=-({}^2v_2/G)C_{11}(C_{14}-C_{44}\xi)V_0, \tag{45}$$

$$ {}^2\alpha_3=-({}^2v_3/G)C_{11}(C_{14}\xi+C_{44})V_0, \tag{46}$$

$$ {}^1\beta_3=-({}^1v_3/G)(C_{14}^2-C_{11}C_{44})(1+\xi^2)V_0, \tag{47}$$

where

$$ G\equiv {}^1v_3(C_{14}^2-C_{11}C_{44})(1+\xi^2)+{}^2v_2C_{11}\,\xi(C_{14}-C_{44}\xi) - {}^2v_3C_{11}(C_{14}\xi+C_{44}), \tag{48}$$

$$ {}^1v_3=(C_{11}/\rho)^{1/2}, \tag{49}$$

$$ {}^2v_2=[(C_{11}+C_{44}-2R)/2\rho]^{1/2}, \tag{50}$$

$$ {}^2v_3=[(C_{11}+C_{44}+2R)/2\rho]^{1/2}, \tag{51}$$

and ξ and R are defined in Table III.

From Eqs. (44)–(47), it is seen that a single pure longitudinal wave propagates back into material 1, and one quasitransverse and one quasilongitudinal wave propagate into material 2. When the two waves in material 2 interact with the free surface, only waves of the same type are reflected back into the sample according to the general result proved at the end of Sec. II. Therefore, the free-surface-velocity history at $x_1'=L$ is given by

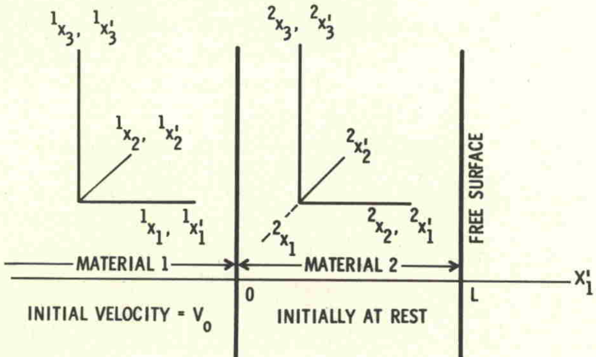

FIG. 4. Geometry and coordinate definitions for impact between x-cut (material 1) and y-cut (material 2) quartz plates. Note that ${}^1x_3'$ and ${}^2x_3'$ are chosen to coincide.

$$ {}^2\dot{\vec{u}}'(L,t)=2\,{}^2\alpha_2\vec{U}^{(2)\prime}H(t-L/{}^2v_2)+2\,{}^2\alpha_3\vec{U}^{(3)\prime}H(t-L/{}^2v_3), \tag{52}$$

for the case of $\eta=0$. The three components of the free-surface velocity as a function of time are shown in Fig. 5. It can be seen here that departures from pure longitudinal wave propagation are significant. This is especially true for $\dot{u}_1'$ and $\dot{u}_3'$. The leading quasilongitudinal wave arrives at $(t/L)=1.67$ µsec/cm and results in large particle velocities in both the x_1' and x_3' directions. This wave is followed by a quasitransverse disturbance which arrives at the free surface at a time given by $(t/L)=2.32$ µsec/cm. This wave also carries changes in $\dot{u}_1'$ and $\dot{u}_3'$.

Large components of particle velocity perpendicular to the propagation direction result in large transverse components of momentum in isolated regions of the impact-loaded sample. Since $\eta=0$ in these calculations, the shear stress transmitted across the impact boundary vanishes and therefore zero transverse momentum is transmitted to the y-cut quartz plate. This requires that the x_2' and x_3' momenta in material 2 must individually sum to zero when contributions from all x_1' positions are considered. The fact that this is indeed the case can be verified from the velocity expression which applies to times before wave interactions occur at the free surface:

$$ {}^2\dot{\vec{u}}(x_1',t)={}^2\alpha_2(-\xi,0,1)H(t-x_1'/{}^2v_2) + {}^2\alpha_3(1,0,\xi)H(t-x_1'/{}^2v_3). \tag{53}$$

The momentum in the x_2' direction clearly vanishes, since ${}^2\dot{u}_2'$ is everywhere zero. The momentum in the x_3' direction per unit area on the impact surface is

$$ \rho({}^2\alpha_2\,{}^2v_2+{}^2\alpha_3\,{}^2v_3\xi) = -(\rho V_0C_{11}/G)[(C_{14}-C_{44}\xi)^2v_2^2 + \xi(C_{14}\xi+C_{44})^2v_3^2], \tag{54}$$

which vanishes identically with the use of Eqs. (50) and (51) and the definitions of ξ and R in Table III. This result does not apply to the case of $\eta\neq 0$ since shear stresses can then be transmitted across the impact surface. However, the total transverse momentum in the target *and* the projectile must always vanish.

Also shown in Fig. 5 is the calculated response for the case of $(1/\eta)=0$: This represents the extreme case in which the two materials become bonded together at the impact surface. It can be seen that conditions at the boundary do not significantly alter $\dot{u}_1'$, but do have an influence on $\dot{u}_3'$ upon arrival of

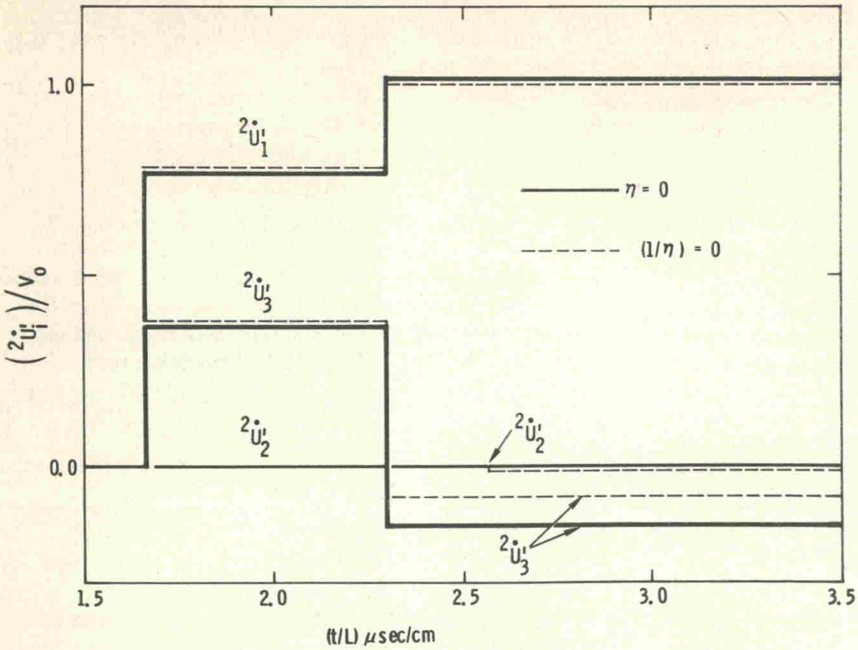

FIG. 5. Free-surface velocity in the y-cut quartz plate as a function of (t/L) for $\eta = 0$ and $(1/\eta) = 0$ in Eqs. (15) and (16).

the quasitransverse wave at $(t/L) = 2.32$ μsec/cm. For $\eta = 0$, $\dot{u}_2'$ remains zero for all times, while for $(1/\eta) = 0$ there is a small negative component of $\dot{u}_2'$ associated with a pure transverse wave which arrives at the free surface at time $(t/L) = 2.58$ μsec/cm.

From the analysis of planar wave propagation in y-cut quartz, it can be seen that multiple-wave effects and transverse motion can be significant. This is in contrast to the results obtained for Be and Ti.

IV. DISCUSSION

A general scheme has been presented for the analysis of plane-wave propagation resulting from planar impact in linearly elastic anisotropic materials of arbitrary crystallographic orientation. The departure from the ideal case of pure longitudinal wave propagation in isotropic materials, as well as in specific directions in anisotropic solids, is manifested by multiple-wave propagation and transverse particle motion. In the case of single-crystal Be and Ti the departure from the ideal situation is small for all crystallographic orientations as seen in Fig. 4: Here $(1 - \sigma_1/\sigma_0)$ is always less than 1%. In fact for $\theta < 15°$ in Be this quantity is less than 10^{-6}. Thus in this region effects resulting from the anisotropic elastic nature of Be are completely negligible. Care must be exercised in extending these conclusions beyond the elastic regime, however. Plastic anisotropy introduced by basal slip in Be may be quite significant; the discussion of such effects requires an analysis different than that presented here.

In certain other materials the effects of elastic anisotropy are not insignificant. Wave propagation produced by planar impact in a y-cut quartz plate can result in large multiple-wave effects and transverse particle motion as shown in Fig. 5. Here it is seen that the longitudinal component of particle velocity associated with the quasitransverse wave is approximately 25% of the final value of $\dot{u}_1'$. It is also seen in this figure that conditions at the impact surface represented by the parameter η in Eqs. (15) and (16) have negligible influence on the longitudinal velocity component, although it does affect the transverse component to a significant degree.

The fact that nonzero transverse momentum may exist locally does not violate the law of momentum conservation when impact is produced by a plate initially moving with a zero transverse velocity component. When the transverse momentum is integrated over the entire physical system, it is found to be zero.

The question naturally arises as to measurement of the effects produced by anisotropic elastic behavior of materials subject to planar impact. In the case of Be and Ti, the effect is probably too small to be observed experimentally even with the most sensitive techniques presently available. If such a measurement were possible, it is likely that only qualitative information on the longitudinal components of stress and velocity could be obtained.

For y-cut quartz, however, the effects are large enough to be observed and measured quantitatively. The laser-interferometer system[22] which measures particle motion in the direction of wave propagation would give values of $\dot{\tilde{u}}_1'$ directly as shown in Fig. 5. Quartz-transducer instrumentation[23,24] should also be able to measure these effects. However, in this latter case the solution for the free surface as obtained in Sec. III would not directly apply and one would have to treat the more complicated interaction of the quasilongitudinal and quasitransverse waves with an x-cut quartz transducer bonded to the y-cut quartz plate at $x_1'=L$ shown in Fig. 4.

Another interesting problem is whether or not it is possible to make a direct measurement on the transverse displacements that may exist under conditions of planar impact. Present techniques have not developed to the point of making such measurements. The results presented here may give some impetus to do so. Also it may be found advantageous to use the tranverse velocity components generated in impact-loaded anisotropic materials to produce large-amplitude compression—shear waves in other solids and thus study their material properties under this more general loading condition.

In conclusion it is seen that plate-impact studies of arbitrarily oriented anisotropic elastic materials can be analyzed in a moderately elementary way. Experimental studies on anisotropic materials in both the elastic and plastic regimes are expected to give significant information on material properties at high strain rates, and the analysis presented here should give a reasonable basis to the mathematical description that will be required.

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